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THE ROTATION OF A RIGID ELLIPSOIDAL INCLUSION EMBEDDED IN AN ANISOTROPIC PIEZOELECTRIC MEDIUM

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Abstract—A rigid ellipsoidal inclusion is embedded in a homogeneous piezoelectric matrix and is rotated infinitesimally, about an axis through its center, by an imposed couple. Without having to solve the governing equations of equilibrium, we find directly the relation between the couple and rotation vectors, together with the stress, strain, rotation tensor, and electric fields just outside the ellipsoidal surface. In addition, we establish boundary integral formulae for evaluation of the fields in the matrix. Gaussian quadrature formulae with variable station points are employed in the numerical computations. Results are presented for a piezoelectric ceramic PZT-6B to show the effect of the aspect ratio of the spheroid on the rotational stiffness. This work extends the results of Walpole (*Proc. R. Soc. London* **A433,** 179–207, 1991) to piezoelectric media.

1. INTRODUCTION

Walpole (1991) recently considered the problem of a rotated rigid ellipsoidal inclusion in an unbounded homogeneous elastic medium using a simple and general "singularity" representation for the elastic fields. Without having to solve either the governing equations of equilibrium in the matrix or the fundamental one of a point force, he was able to find the relation between the couple and rotation vectors, and the components of stress, strain and rotation at points just outside the inclusion. The objective of this work is to extend Walpole's approach to piezoelectric media.

Piezoelectric materials exhibit coupling behavior between mechanical and electric fields and are inherently anisotropic. Exact solutions of boundary value problems in such media are rather scarce in the literature. We consider a rigid ellipsoidal inclusion embedded in a homogeneous, arbitrarily anisotropic, piezoelectric matrix and rotated infinitesimally, about an axis through its center, by an imposed couple. The term "rigid" is defined here in the sense that its stiffness and dielectric permittivity tend to infinity so that no elastic strain or electric field is present in the inclusion. The approach is to let the homogeneous piezoelectric medium extend throughout the whole space. A layer of body force and charge is introduced over the ellipsoidal surface at a density that has the *constant* linear combinations of the outward unit normal. By a suitable choice of one first- and two second-rank coefficients, we reproduce a uniform rotation tensor to the interior, but not accompanied by any strain or electric field, while the exterior elastic and electric fields are identical to those associated with the rotated rigid inclusion. The concept is close to that devised by Eshelby (1957) and the formulation is related to the subject of interfacial discontinuities (Hill, 1983; Chen, 1993a). We find directly the relation between the couple and rotation vectors, together with the stress, strain, rotation, electric field and electric displacement just outside the inclusion. Moreover, we establish boundary integral formulae for evaluation of the fields in the matrix. In the case of a uniform strain and/or a uniform intensity applied at infinity, we can superimpose uniform fields of strain and intensity in the medium to obtain solutions. The results are expressed in a closed form and are evaluated numerically without regard to the anisotropy of the medium or to the ellipticity of the inclusion. Gaussian quadrature formulae with a variable number of integration points are employed in the calculations. The computer routines have been checked with existing analytic solutions for purely elastic cases. As an illustration, we present results for a piezoelectric ceramic PZT-6B to show the effect of the aspect ratio of the spheroid on the rotational stiffness.

Available solutions of problems of this kind in elasticity can be found in the papers by Kanwal and Sharma (1976), Selvadurai (1980, 1984), and Zureick and Choi (1989). Kanwal and Sharma applied the singularity method to obtain the displacement field for a general rotation and translation of a rigid prolate or oblate spheroidal inclusion in an isotropic matrix. Selvadurai employed the Hankel integral transform to investigate the asymmetric displacement of a rigid elliptical disc in a transversely isotropic medium. Zureick and Choi studied the rotation of a rigid spheroidal inclusion embedded in a transversely isotropic medium using the displacement potential method.

We start with a review of basic equations in Section 2. The essence of the method is decribed in Section 3 through the formulation of an ellipsoidal layer of body force and charge in an unbounded piezoelectric medium. We prove some related theorems in Section 4 with the help of reciprocal relations and establish our main results in Section 5. Finally, numerical results are presented in Section 6. Cartesian tensors will be used and their components will be written by the indicial notation, with reference to the coordinates x_1, x_2, x_3. Repeated indices indicate Einstein's summation convention with the index running from 1 to 3. $\mathbf{i}$ is the unit second-rank tensor δ_{ij} such that $\boldsymbol{\alpha}\boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^{-1}\boldsymbol{\alpha} = \mathbf{i}$, provided that $\boldsymbol{\alpha}$ is invertible.

2. BASIC EQUATIONS

The constitutive relation for a linear piezoelectric medium can be expressed as (Tiersten, 1969):

$$\begin{cases} \sigma_{ij} = L_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \\ D_i = e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k, \end{cases} \tag{1}$$

where σ_{ij} is the stress tensor, ε_{ij} the strain tensor, D_i the electric displacement vector, and E_i the electric intensity. L_{ijkl} are the elastic moduli measured in a constant electric field; κ_{ij} are the dielectric permittivities measured at constant strain; e_{ijk} are the piezoelectric constants. The material constants $\mathbf{L}$, $\mathbf{e}$, $\boldsymbol{\kappa}$ are, respectively, fourth-rank, third-rank, and second-rank tensors, which satisfy the symmetry relations:

$$L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}, \quad e_{ikl} = e_{ilk}, \quad \kappa_{ij} = \kappa_{ji}, \tag{2}$$

so that L_{ijkl}, e_{ijk} and κ_{ij} admit, at most, 21, 18 and 6 independent components, respectively. If $u_i(\mathbf{x})$ is the elastic displacement vector and $\phi(\mathbf{x})$ the electric potential, the infinitesimal strain, rotation tensor and electric field are given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}), \quad E_i = -\phi_{,i}, \tag{3}$$

where the comma followed by an index indicates the derivative with respect to the corresponding space coordinate. The stress and electric displacement should satisfy the divergence equations

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \tag{4}$$

where the body forces and the extrinsic charges are neglected.

3. AN ELLIPSOIDAL LAYER OF BODY FORCE AND CHARGE

An unbounded volume of a homogeneous piezoelectric medium is loaded by a layer of body force and charge over a closed surface S. At the remote boundary of the matrix the strain and electric field are zero. The resulting displacement field and electric potential are hence continuous across S and continuously differentiable elsewhere. The interfacial jumps in the surface traction and charge are specified at a density that has constant linear combinations of the components of the outward unit normal **n** on S, that is

$$(\sigma_{ij}^{\mathrm{I}}-\sigma_{ij}^{\mathrm{E}})n_j = (t_{ij}+\tau_{ij})n_j, \quad (D_i^{\mathrm{I}}-D_i^{\mathrm{E}})n_i = -q_i n_i, \tag{5}$$

in which the general unsymmetric set of constant second-rank coefficients is separated into symmetric and antisymmetric parts, i.e. $t_{ij} = t_{ji}$, $\tau_{ij} = -\tau_{ji}$. The superscripts I and E, respectively, refer to the interior and exterior parts of S. Since there is no body force or charge elsewhere, the equilibrium conditions (4) are satisfied at points inside and ouside S. The resultant force, couple and charge due to the point layer of force and charge could be evaluated by the surface integrals

$$\int (t_{ij}+\tau_{ij})n_j\,\mathrm{d}S = 0, \quad \int \epsilon_{ijk}x_j(t_{kl}+\tau_{kl})n_l\,\mathrm{d}S = V\epsilon_{ijk}\tau_{kj}, \quad \int q_i n_i\,\mathrm{d}S = 0, \tag{6}$$

by appeal to the divergence theorem, where V is the volume enclosed by S and ϵ_{ijk} is the permutation symbol. Equation (6) indicates that the resultant force and charge are zero, but the couple is generally not.

To find the jump relations on both sides of S, we start with the Hadamard's geometric interpretation for the displacement gradient and electric field (Hill, 1961)

$$\begin{aligned} u_{i,j}^{\mathrm{I}}-u_{i,j}^{\mathrm{E}} &= \xi_i n_j, \\ \phi_{,i}^{\mathrm{I}}-\phi_{,i}^{\mathrm{E}} &= h n_i. \end{aligned} \tag{7}$$

Substituting (7) into (1) we obtain

$$\begin{aligned} (\sigma_{ij}^{\mathrm{I}}-\sigma_{ij}^{\mathrm{E}})n_j &= c_{ik}\xi_k + d_i h, \\ (D_i^{\mathrm{I}}-D_i^{\mathrm{E}})n_i &= d_i\xi_i - ph, \end{aligned} \tag{8}$$

where

$$c_{ik} = L_{ijkl}n_j n_l, \quad d_j = e_{ijk}n_i n_k, \quad p = \kappa_{ij}n_i n_j. \tag{9}$$

The tensor ξ_i and scalar h are some unknowns that could be determined from (5) and (8) as:

$$\begin{aligned} \xi_i &= \tfrac{1}{2}(k_{ik}n_l+k_{il}n_k)t_{kl}+\tfrac{1}{2}(k_{ik}n_l-k_{il}n_k)\tau_{kl}-\frac{1}{p}k_{ik}d_k q_j n_j, \\ h &= \frac{1}{2}\frac{1}{p}d_j[(k_{jl}n_k+k_{jk}n_l)t_{kl}+\tfrac{1}{2}(k_{jk}n_l-k_{jl}n_k)\tau_{kl}]-\frac{1}{p}\left(\frac{1}{p}k_{mm}d_m d_n-1\right)q_k n_k, \end{aligned} \tag{10}$$

where $k_{ij} = (c_{ij}+(1/p)d_i d_j)^{-1}$. By some tensorial algebra it can also be shown that k_{ij} is equivalent to

$$k_{ij} = c_{ij}^{-1}-\frac{1}{\chi}c_{ik}^{-1}c_{il}^{-1}d_k d_l, \quad \chi = c_{pq}^{-1}d_p d_q. \tag{11}$$

Substituting (10) into (7) and taking the symmetric and antisymmetric parts in (7_1), we find the jump relations

$$\begin{aligned} \varepsilon_{ij}^{\mathrm{I}}-\varepsilon_{ij}^{\mathrm{E}} &= \mathscr{P}_{ijkl}t_{kl}+\mathscr{R}_{ijkl}\tau_{kl}+\mathscr{S}_{ijk}q_k, \\ \omega_{ij}^{\mathrm{I}}-\omega_{ij}^{\mathrm{E}} &= \mathscr{R}_{klij}t_{kl}+\mathscr{Q}_{ijkl}\tau_{kl}+\mathscr{T}_{ijk}q_k, \end{aligned}$$

$$E_i^{\rm I}-E_i^{\rm E} = \mathscr{S}_{kli}t_{kl}+\mathscr{T}_{kli}\tau_{kl}+\mathscr{U}_{ik}q_k, \tag{12}$$

for the strain, rotation and electric fields at points of S, where the tensorial coefficients are of the form

$$\begin{aligned}
\mathscr{P}_{ijkl} &= \tfrac{1}{4}(k_{ik}n_jn_l+k_{il}n_jn_k+k_{jk}n_in_l+k_{jl}n_in_k),\\
\mathscr{Q}_{ijkl} &= \tfrac{1}{4}(k_{ik}n_jn_l-k_{il}n_jn_k-k_{jk}n_in_l+k_{jl}n_in_k),\\
\mathscr{R}_{ijkl} &= \tfrac{1}{4}(k_{ik}n_jn_l-k_{il}n_jn_k+k_{jk}n_in_l-k_{jl}n_in_k),\\
\mathscr{S}_{ijk} &= -\frac{1}{2}\frac{1}{p}d_l(k_{il}n_jn_k+k_{jl}n_in_k),\\
\mathscr{T}_{ijk} &= -\frac{1}{2}\frac{1}{p}d_l(k_{il}n_jn_k-k_{jl}n_in_k),\\
\mathscr{U}_{ik} &= \frac{1}{p}\left(\frac{1}{p}k_{mn}d_md_n-1\right)n_in_k.
\end{aligned} \tag{13}$$

Since k_{ij} is symmetric these operators satisfy the conditions

$$\begin{aligned}
&\mathscr{P}_{ijkl} = \mathscr{P}_{jikl} = \mathscr{P}_{ijlk} = \mathscr{P}_{klij}, \quad \mathscr{Q}_{ijkl} = -\mathscr{Q}_{jikl} = -\mathscr{Q}_{ijlk} = \mathscr{Q}_{klij},\\
&\mathscr{R}_{ijkl} = \mathscr{R}_{jikl} = -\mathscr{R}_{ijlk}, \quad \mathscr{S}_{ijk} = \mathscr{S}_{jik}, \quad \mathscr{T}_{ijk} = -\mathscr{T}_{jik}, \quad \mathscr{U}_{ij} = \mathscr{U}_{ji}.
\end{aligned} \tag{14}$$

In the next section we shall show that if S is an ellipsoidal shape, the strain, rotation tensor and electric field are uniform inside the ellipsoid, and the following integrals vanish over S

$$\int_S w\varepsilon_{ij}^{\rm E}\,{\rm d}S = 0, \quad \int_S w\omega_{ij}^{\rm E}\,{\rm d}S = 0, \quad \int_S wE_i^{\rm E}\,{\rm d}S = 0, \tag{15}$$

where the weighting function $w(x)$ is the perpendicular distance from the origin to the tangent plane of the ellipsoid at each point, namely $w = x_rn_r$.

Now multiply both sides of (12) by $w(x)$ and integrate over the ellipsoidal surface S; by use of (15) we obtain the interior uniform strain, rotation and electric fields as

$$\begin{aligned}
\varepsilon_{ij}^{\rm I} &= P_{ijkl}t_{kl}+R_{ijkl}\tau_{kl}+S_{ijk}q_k,\\
\omega_{ij}^{\rm I} &= R_{klij}t_{kl}+Q_{ijkl}\tau_{kl}+T_{ijk}q_k,\\
E_i^{\rm I} &= S_{kli}t_{kl}+T_{kli}\tau_{kl}+U_{ik}q_k,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
P_{ijkl} &= \frac{1}{4\pi a_1a_2a_3}\int w\mathscr{P}_{ijkl}\,{\rm d}S, \quad Q_{ijkl} = \frac{1}{4\pi a_1a_2a_3}\int w\mathscr{Q}_{ijkl}\,{\rm d}S,\\
R_{ijkl} &= \frac{1}{4\pi a_1a_2a_3}\int w\mathscr{R}_{ijkl}\,{\rm d}S, \quad S_{ijk} = \frac{1}{4\pi a_1a_2a_3}\int w\mathscr{S}_{ijk}\,{\rm d}S,\\
T_{ijk} &= \frac{1}{4\pi a_1a_2a_3}\int w\mathscr{T}_{ijk}\,{\rm d}S, \quad U_{ij} = \frac{1}{4\pi a_1a_2a_3}\int w\mathscr{U}_{ij}\,{\rm d}S,
\end{aligned} \tag{17}$$

and where a_1, a_2 and a_3 are the semi-axes of the ellipsoidal surface S.

4. AN ELLIPSOIDAL VOLUME DISTRIBUTION OF BODY FORCE AND CHARGE

We shall show in this section that the elastic strain, rotation tensor and electric field are uniform inside the ellipsoidal region under an ellipsoidal layer of body force and charge as described in Section 3. To prove this, we employ the approach of Green's function. Similar to that of elasticity, Green's functions in piezoelectric media can be defined as [see for example, Minagawa (1984)]:

$$\begin{bmatrix} L_{ijkl}\dfrac{\partial^2}{\partial x_j\,\partial x_l} & e_{kij}\dfrac{\partial^2}{\partial x_j\,\partial x_k} \\ e_{ikl}\dfrac{\partial^2}{\partial x_i\,\partial x_l} & -\kappa_{ik}\dfrac{\partial^2}{\partial x_i\,\partial x_k} \end{bmatrix} \begin{bmatrix} g^1_{kp} & g^2_k \\ g^3_p & g^4 \end{bmatrix} = \begin{bmatrix} -\delta_{ip}\delta(\mathbf{x}-\mathbf{x}') & 0 \\ 0 & -\delta(\mathbf{x}-\mathbf{x}') \end{bmatrix}, \tag{18}$$

where δ_{ij} is the Kronecker delta and $\delta(\mathbf{x}-\mathbf{x}')$ the Dirac delta function. $g^1_{ij}(\mathbf{x}-\mathbf{x}')$ and $g^3_j(\mathbf{x}-\mathbf{x}')$ are, respectively, defined to be the elastic displacement in the i direction and electric potential at $\mathbf{x}$ due to a point force applied at $\mathbf{x}'$ in the x_j direction; likewise g^2_i and g^4 are, respectively, the displacement in the i direction and electric potential at $\mathbf{x}$ due to a point charge at $\mathbf{x}'$.

Now consider a uniform distribution of body force and charge acting on the region inside a closed surface S. The corresponding equations for the displacement and electric potential can be written as

$$\begin{bmatrix} L_{ijkl}\dfrac{\partial^2}{\partial x_j\,\partial x_l} & e_{kij}\dfrac{\partial^2}{\partial x_j\,\partial x_k} \\ e_{ikl}\dfrac{\partial^2}{\partial x_i\,\partial x_l} & -\kappa_{ik}\dfrac{\partial^2}{\partial x_i\,\partial x_k} \end{bmatrix} \begin{bmatrix} G^1_{kp} & G^2_k \\ G^3_p & G^4 \end{bmatrix} = \begin{bmatrix} -\delta_{ip}l(\mathbf{x}) & 0 \\ 0 & l(\mathbf{x}) \end{bmatrix}, \tag{19}$$

where

$$l(\mathbf{x}) = \begin{cases} 1, & \mathbf{x}\in V \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

To establish the relation between the tensors $\mathbf{G}$ and $\mathbf{g}$, we introduce the following reciprocal relation

$$\left\{\begin{bmatrix} L_{ijkl} & e_{kij} \\ e_{ikl} & -\kappa_{ik} \end{bmatrix} \begin{bmatrix} \dfrac{\partial G^1_{km}}{\partial x_l}F_m & \dfrac{\partial G^2_k}{\partial x_l}\phi \\ \dfrac{\partial G^3_m}{\partial x_k}F_m & \dfrac{\partial G^4}{\partial x_k}\phi \end{bmatrix}\right\} \begin{bmatrix} \dfrac{\partial g^1_{in}}{\partial x_j}F'_n & \dfrac{\partial g^2_i}{\partial x_j}\phi' \\ \dfrac{\partial g^3_n}{\partial x_i}F'_n & \dfrac{\partial g^4}{\partial x_i}\phi' \end{bmatrix}$$

$$= \left\{\begin{bmatrix} L_{ijkl} & e_{kij} \\ e_{ikl} & -\kappa_{ik} \end{bmatrix} \begin{bmatrix} \dfrac{\partial g^1_{km}}{\partial x_l}F'_m & \dfrac{\partial g^2_k}{\partial x_l}\phi' \\ \dfrac{\partial g^3_m}{\partial x_k}F'_m & \dfrac{\partial g^4}{\partial x_k}\phi' \end{bmatrix}\right\} \begin{bmatrix} \dfrac{\partial G^1_{in}}{\partial x_j}F_n & \dfrac{\partial G^2_i}{\partial x_j}\phi \\ \dfrac{\partial G^3_n}{\partial x_i}F_n & \dfrac{\partial G^4}{\partial x_i}\phi \end{bmatrix}. \tag{21}$$

This equation is valid for the entire medium due to the diagonal symmetry of the moduli and the bracketed terms are placed to indicate the components of stress and electric displacement. Integrating (21) over the interior and exterior of S by appeal to the divergence theorem and to the equilibrium equations, one can find the functions $\mathbf{G}$ as

$$G^1_{ij}(\mathbf{x}) = \int_V g^1_{ji}(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}', \quad G^4(\mathbf{x}) = \int_V g^4(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}',$$

$$G^2_i(\mathbf{x}) = G^3_i(\mathbf{x}) = \int_V g^2_i(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}' = \int_V g^3_i(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}'. \tag{22}$$

It should be noted that, similar to that of elasticity, the Green functions of piezoelectric media possess the decomposition (Chen, 1993b)

$$\mathbf{g} = \hat{\mathbf{g}}(\mathbf{S})/|\mathbf{x}-\mathbf{x}'|, \quad S_i = (x_i - x_i')/|\mathbf{x}-\mathbf{x}'|, \tag{23}$$

where $\hat{\mathbf{g}}$ are even functions of $\mathbf{S}$. With this property one can show that the volume integrations (22) are simply quadratic functions of the coordinates

$$G^1_{ij} = \tfrac{1}{2}(J^1_{ij} - K^1_{ijkl}x_k x_l), \quad G^2_i = \tfrac{1}{2}(J^2_i - K^2_{ikl}x_k x_l),$$
$$G^3_i = \tfrac{1}{2}(J^3_i - K^3_{ikl}x_k x_l), \quad G^4 = \tfrac{1}{2}(J^4 - K^4_{kl}x_k x_l), \tag{24}$$

for points interior to S, where $\mathbf{J}$ and $\mathbf{K}$ are some constant tensorial coefficients. At points far away from the region V the tensors $\mathbf{G}$ are asymptotically of the order r^{-1}. It is also obvious that these coefficients possess the symmetry properties

$$J^1_{ij} = J^1_{ji}, \quad K^1_{ijkl} = K^1_{jikl} = K^1_{ijlk}, \quad K^3_{ikl} = K^3_{ilk}, \quad K^4_{kl} = K^4_{lk}. \tag{25}$$

Since there are no discontinuities in the displacement, potential, traction or in the electric displacement across the surface S, it may be inferred that the functions $\mathbf{G}$ and their first derivatives are continuous. However, their second derivatives may have the jumps (Hill, 1961)

$$G^{1_E}_{kp,jl} - G^{1_I}_{kp,jl} = \lambda_{kp}n_j n_l, \quad G^{3_E}_{p,jk} - G^{3_I}_{p,jk} = \alpha_p n_j n_k,$$
$$G^{2_E}_{k,il} - G^{2_I}_{k,il} = \beta_k n_i n_l, \quad G^{4_E}_{,jl} - G^{4_I}_{,jl} = \gamma n_j n_l, \tag{26}$$

$\boldsymbol{\lambda}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and γ being some tensors and a scalar. In accordance with (19), multiplying (26) by two suitable members of $\mathbf{L}$, $\mathbf{e}$, or $\boldsymbol{\kappa}$ and subtracting these two equations, we can solve for the unknowns as

$$\lambda_{kp} = k_{kp}, \quad \alpha_k = \beta_k = \frac{1}{p}k_{ki}d_i, \quad \gamma = \frac{1}{p}\left(\frac{1}{p}k_{mn}d_m d_n - 1\right). \tag{27}$$

We shall show in the Appendix that the weighted averages of the surface integrals vanish

$$\int_S wG^{1_E}_{ij,kl}\,\mathrm{d}S = 0, \quad \int_S wG^{2_E}_{i,kl}\,\mathrm{d}S = 0,$$

$$\int_S wG^{3_E}_{i,kl}\,\mathrm{d}S = 0, \quad \int_S wG^{4_E}_{i,kl}\,\mathrm{d}S = 0. \tag{28}$$

Hence the immediate connections from (24), (26–28) are

$$K^1_{ijkl} = \left(\int wk_{ij}n_kn_l\,\mathrm{d}S\right)\bigg/\left(\int w\,\mathrm{d}S\right),$$

$$K^2_{ikl} = K^3_{ikl} = \left(\int w\frac{1}{p}k_{ij}d_jn_kn_l\,\mathrm{d}S\right)\bigg/\left(\int w\,\mathrm{d}S\right),$$

$$K^4_{ij} = \left(\int w\frac{1}{p}\left[\frac{1}{p}k_{mn}d_md_n - 1\right]n_in_j\,\mathrm{d}s\right)\bigg/\left(\int w\,\mathrm{d}S\right). \tag{29}$$

Now returning to the problem of an ellipsoidal layer of body force and charge (5), the full elastic and electric fields can be constructed from the Green function (18) by appeal to the reciprocal relation (21) as

$$u_i = \int_S (t_{jk}+\tau_{jk})n_kg^1_{ji}\,\mathrm{d}S - \int_S (q_kn_k)g^2_i\,\mathrm{d}S,$$

$$\phi = \int_S (t_{jk}+\tau_{jk})n_kg^3_j\,\mathrm{d}S - \int_S (q_kn_k)g^4\,\mathrm{d}S. \tag{30}$$

It is convenient to convert the integration into the interior volume by use of the divergence theorem

$$\begin{aligned} u_i &= -\tfrac{1}{2}t_{jk}(G^1_{ij,k}+G^1_{ik,j}) - \tfrac{1}{2}\tau_{jk}(G^1_{ij,k}-G^1_{ik,j}) + q_kG^2_{i,k}, \\ \phi &= -\tfrac{1}{2}t_{jk}(G^3_{j,k}+G^3_{k,j}) - \tfrac{1}{2}\tau_{jk}(G^3_{j,k}-G^3_{k,j}) + q_kG^4_{,k}. \end{aligned} \tag{31}$$

Since G^1_{ij}, G^2_i, G^3_i and G^4 are quadratic functions of x_i inside the ellipsoid, the interior strain $\varepsilon^{\mathrm{I}}_{ij}$, rotation ω^{I}_{ij} and electric field E^{I}_i are then constant. We thus complete the proof. In addition, eqn (16) is recovered again by substituting (31) and (29) into (3).

5. A ROTATED RIGID ELLIPSOIDAL INCLUSION

Consider a rigid ellipsoidal inclusion firmly embedded in an infinte piezoelectric medium and subjected to a prescribed couple or a rotation vector about an axis through its center. The term "rigid" is defined here in the sense that its stiffness and dielectric permittivity tend to infinity so that no elastic strain or electric field is present in the inclusion. We intend to reproduce the response of a rotated rigid inclusion by prescribing an ellipsoidal layer of body force and charge as stated in Section 3. Physically, in the rotated rigid inclusion, the elastic strain and electric field are vanishing, and the rotation tensor is constant. To find the unknown coefficients t_{ij}, τ_{ij} and q_i, we first invert eqn (16) as

$$\begin{aligned} t_{ij} &= P^*_{ijkl}\varepsilon^{\mathrm{I}}_{kl} + R^*_{ijkl}\omega^{\mathrm{I}}_{kl} + S^*_{ijk}E^{\mathrm{I}}_k, \\ \tau_{ij} &= R^*_{klij}\varepsilon^{\mathrm{I}}_{kl} + Q^*_{ijkl}\omega^{\mathrm{I}}_{kl} + T^*_{ijk}E^{\mathrm{I}}_k, \\ q_i &= S^*_{kli}\varepsilon^{\mathrm{I}}_{kl} + T^*_{kli}\omega^{\mathrm{I}}_{kl} + U^*_{ik}E^{\mathrm{I}}_k, \end{aligned} \tag{32}$$

and consequently obtain the results as

$$t_{ij} = R^*_{ijkl}\omega^{\mathrm{I}}_{kl}, \quad \tau_{ij} = Q^*_{ijkl}\omega^{\mathrm{I}}_{kl}, \quad q_i = T^*_{kli}\omega^{\mathrm{I}}_{kl}. \tag{33}$$

Accordingly the displacement and potential have the components

$$u_i = \omega^{\mathrm{I}}_{ij} x_j = \epsilon_{ijk} \Omega^{\mathrm{I}}_j x_k, \quad \phi = 0, \tag{34}$$

at interior and immediate exterior of S, where the corresponding axial rotation vector Ω^{I}_i is of the form

$$\Omega^{\mathrm{I}}_i = -\tfrac{1}{2}\epsilon_{ijk}\omega^{\mathrm{I}}_{jk}, \quad \omega^{\mathrm{I}}_{ij} = -\epsilon_{ijk}\Omega^{\mathrm{I}}_k. \tag{35}$$

Now the interior and exterior fields of displacement, potential, strain and electric field, and the exterior field of stress and electric displacement are exactly identical to those associated with the rotated rigid ellipsoid. Since σ^{I}_{ij} vanishes in the replacement configuration, the resultant couple can be evaluated by the surface integration over S with the help of (5) and (6)

$$\Gamma_i = \int_S \epsilon_{ijk} x_j \sigma^{\mathrm{E}}_{kl} n_l \, \mathrm{d}S = V\epsilon_{ijk}\tau_{jk} = -2Vb^*_{ij}\Omega^{\mathrm{I}}_j, \tag{36}$$

in which

$$b^*_{ij} = \tfrac{1}{2}\epsilon_{imn}\epsilon_{jpq} Q^*_{mnpq} = Q^*_{mnmn}\delta_{ij} - 2Q^*_{mjmi}. \tag{37}$$

Thus for a prescribed couple Γ_i we can readily determine the rotation tensor ω^{I}_{ij} and hence the quantities t_{ij}, τ_{ij} and q_i. The matrix interfacial quantities can be derived as

$$\begin{aligned}
\varepsilon^{\mathrm{E}}_{ij} &= -\mathscr{P}_{ijkl} t_{kl} - \mathscr{R}_{ijkl}\tau_{kl} - \mathscr{S}_{ijk} q_k, \\
\omega^{\mathrm{E}}_{ij} &= \omega^{\mathrm{I}}_{ij} - \mathscr{R}_{klij} t_{kl} - \mathscr{Q}_{ijkl}\tau_{kl} - \mathscr{T}_{ijk} q_k, \\
E^{\mathrm{E}}_i &= -\mathscr{S}_{kli} t_{kl} - \mathscr{T}_{kli}\tau_{kl} - \mathscr{U}_{ik} q_k,
\end{aligned} \tag{38}$$

and hence the corresponding stress and electric displacement.

In the case of a uniform strain and electric field applied at infinity, we superimpose uniform fields of strain $\varepsilon^{\mathrm{A}}_{ij}$, rotation ω^{A}_{ij}, and intensity E^{A}_i (and hence stress σ^{A}_{ij} and electric displacement D^{A}_i) in the unbounded homogeneous medium. Now stipulating

$$\varepsilon^{\mathrm{I}}_{ij} = -\varepsilon^{\mathrm{A}}_{ij}, \quad \omega^{\mathrm{I}}_{ij} = \omega^{\mathrm{B}}_{ij} - \omega^{\mathrm{A}}_{ij}, \quad E^{\mathrm{I}}_i = -E^{\mathrm{A}}_i, \tag{39}$$

to eliminate the interior strain and intensity, we find the unknown coefficients as

$$\begin{aligned}
t_{ij} &= -P^*_{ijkl}\varepsilon^{\mathrm{A}}_{kl} + R^*_{ijkl}(\omega^{\mathrm{B}}_{kl} - \omega^{\mathrm{A}}_{kl}) - S^*_{ijk}E^{\mathrm{A}}_k, \\
\tau_{ij} &= -R^*_{klij}\varepsilon^{\mathrm{A}}_{kl} + Q^*_{ijkl}(\omega^{\mathrm{B}}_{kl} - \omega^{\mathrm{A}}_{kl}) - T^*_{ijk}E^{\mathrm{A}}_k, \\
q_i &= -S^*_{kli}\varepsilon^{\mathrm{A}}_{kl} + T^*_{kli}(\omega^{\mathrm{B}}_{kl} - \omega^{\mathrm{A}}_{kl}) - U^*_{ik}E^{\mathrm{A}}_k,
\end{aligned} \tag{40}$$

where ω^{B}_{ij} is the total interior rotation tensor externally imposed or induced from the applied couple Γ^{B}_i. By means of the surface integration analogous to (36), we find the couple–rotation relationship for the inclusion

$$\begin{aligned}
\Gamma^{\mathrm{B}}_i/V &= \frac{1}{V}\int \epsilon_{ijk} x_j(\sigma^{\mathrm{E}}_{kl} + \tau^{\mathrm{A}}_{kl}) n_l \, \mathrm{d}S = \epsilon_{ijk}\tau_{jk} \\
&= -\epsilon_{ijk} R^*_{lmjk}\varepsilon^{\mathrm{A}}_{lm} - 2b^*_{ij}(\Omega^{\mathrm{B}}_j - \Omega^{\mathrm{A}}_j) - \epsilon_{ijk} T^*_{jkl} E^{\mathrm{A}}_l.
\end{aligned} \tag{41}$$

The interfacial quantities on the matrix side are thus

$$\begin{aligned}
\varepsilon^{\mathrm{E}}_{ij} &= -\varepsilon^{\mathrm{A}}_{ij} - \mathscr{P}_{ijkl} t_{kl} - \mathscr{R}_{ijkl}\tau_{kl} - \mathscr{S}_{ijk} q_k, \\
\omega^{\mathrm{E}}_{ij} &= -\omega^{\mathrm{A}}_{ij} + \omega^{\mathrm{B}}_{ij} - \mathscr{R}_{klij} t_{kl} - \mathscr{Q}_{ijkl}\tau_{kl} - \mathscr{T}_{ijk} q_k,
\end{aligned}$$

$$E_i^{\mathrm{E}} = -E_i^{\mathrm{A}} - \mathscr{S}_{kli} t_{kl} - \mathscr{T}_{kli} \tau_{kl} - \mathscr{U}_{ik} q_k, \tag{42}$$

and hence one can derive the stress and electric displacement.

We have now obtained the interfacial quantities just outside the inclusion. With this information we shall further pursue the fields at points inside the matrix. Specifically, the problem can be treated as an infinite piezoelectric medium with an ellipsoidal cavity inside. On its surface the stress, strain, electric field and electric displacements are all given, while at its remote boundary the strain and electric fields are either zero or uniform. Our objective is to derive some formulae for evaluation of the fields in the matrix. Due to the symmetric nature of $L_{ijkl} = L_{klij}$ and $\kappa_{ij} = \kappa_{ji}$ we can establish the following reciprocal relation

$$\int_V (\sigma_{ij} g^1_{ip,j} + D_i g^3_{p,i})\,\mathrm{d}V = \int_V (L_{ijkl} g^1_{kp,l} + e_{kij} g^3_{p,k}) u_{i,j}\,\mathrm{d}V + \int_V (e_{ikl} g^1_{kp,l} - \kappa_{ik} g^3_{p,k})\phi_{,i}\,\mathrm{d}V. \tag{43}$$

Now invoking the divergence theorem and recalling the definition of Green's function, after some reduction we can show that

$$u_p(\mathbf{x}) = \varepsilon^{\mathrm{A}}_{pm} x_m + \int_S \sigma_{ij} n_j g^1_{ip}\,\mathrm{d}S + \int_S D_i n_i g^3_p\,\mathrm{d}S - \int_S u_i (L_{ijkl} g^1_{kp,l} + e_{kij} g^3_{p,k}) n_j\,\mathrm{d}S - \int_S \phi (e_{ikl} g^1_{kp,l} - \kappa_{ik} g^3_{p,k}) n_i\,\mathrm{d}S, \tag{44}$$

where the boundary terms at infinity are reduced to the first member on the right-hand side of (44). Analogous to (43), we can also write the reciprocal relation

$$\int_V (\sigma_{ij} g^2_{i,j} + D_i g^4_{,i})\,\mathrm{d}V = \int_V (L_{ijkl} g^2_{k,l} + e_{kij} g^4_{,k}) u_{i,j}\,\mathrm{d}V + \int_V (e_{ikl} g^2_{k,l} - \kappa_{ik} g^4_{,k})\phi_{,i}\,\mathrm{d}V, \tag{45}$$

and consequently establish that

$$\phi(\mathbf{x}) = \phi^{\mathrm{A}}_{,m} x_m + \int_S \sigma_{ij} n_j g^2_i\,\mathrm{d}S + \int_S D_i n_i g^4\,\mathrm{d}S - \int_S u_i (L_{ijkl} g^2_{k,l} + e_{kij} g^4_{,k}) n_j\,\mathrm{d}S - \int_S \phi (e_{ikl} g^2_{k,l} - \kappa_{ik} g^4_{,k}) n_i\,\mathrm{d}S. \tag{46}$$

We mention that a numerical evaluation procedure for the Green's functions and their first derivatives are outlined by Chen (1993b). Since all field variables on S are known, eqns (44) and (46) are readily amenable to numerical computations. In the context of elasticity the corresponding formula for (44) and (46) is the well-known Somigliana identity (Love, 1944) which constitutes the basic formulation in the direct method of boundary element.

6. NUMERICAL RESULTS

As seen from the previous section it is obvious that the solutions rely on the evaluations of the integrals (17). Unfortunately, for arbitrary anisotropy of the medium and for arbitrary ellipticity of the inclusion it is not generally possible to obtain the results analytically. Thus we shall carry out the integrations numerically in terms of Gaussian double quadratures. In order to do this, the integrals (17) are first parameterized on the surface of a unit sphere following a coordinate transformation described in Mura (1987). For example, the tensor P_{ijkl} (17_1) can be written as

$$P_{ijkl} = \frac{1}{16\pi}\int_{-1}^{1} \mathrm{d}\zeta_3 \int_0^{2\pi} (k_{ik}\xi_j\xi_l + k_{il}\xi_j\xi_k + k_{jk}\xi_i\xi_l + k_{jl}\xi_i\xi_k)\,\mathrm{d}\omega, \tag{47}$$

where

$$k_{ik} = k_{ik}(\xi), \quad \xi_i = \zeta_i/a_i, \quad \text{(no sum on } i\text{)},$$
$$\zeta_1 = (1-\zeta_3^2)^{1/2}\cos\omega, \quad \zeta_2 = (1-\zeta_3^2)^{1/2}\sin\omega, \tag{48}$$

and similarly for **Q**, **R** and others. The double integration (47) can be computed using the Gaussian quadrature formula

$$P_{ijkl} = \frac{1}{16\pi}\sum_{p=1}^{M}\sum_{q=1}^{N} [(k_{ik}\xi_j\xi_l + k_{il}\xi_j\xi_k + k_{jk}\xi_i\xi_l + k_{jl}\xi_i\xi_k)W_{pq}(\omega_q, \zeta_{3p})], \tag{49}$$

where $\xi_i = \xi_i(\omega_q, \zeta_{3p})$, M and N refer to the Gaussian points used for the integration over ζ_3 and ω, respectively, and W_{pq} are the Gaussian weights. The constants M and N are variable numbers depending on the aspect ratio of the ellipsoid, material constants, and the desired accuracy. We mention the work of Ghahremani (1977) who used a different parameterization of the unit sphere for evaluations of **P** tensor in elasticity.

In order to perform the numerical inverse (32), it is convenient to write (16) in a matrix notation. This can be achieved by using the following convention: replace the first two suffixes by a single one and/or the last two in the same way according to the following rule

ij or kl	11	22	33	23, 32	31, 13	12, 21
m or n	1	2	3	4	5	6

(50)

for the stress and strain, and

ij or kl	32	13	21
s or t	1	2	3

(51)

for the antisymmetric tensors τ and ω. Accordingly eqn (16) can be written in the form:

$$\begin{aligned} \boldsymbol{\varepsilon}^{\mathrm{I}} &= \mathbf{Pt} + \mathbf{R}\boldsymbol{\tau} + \mathbf{Sq}, \\ \boldsymbol{\omega}^{\mathrm{I}} &= \mathbf{R}^{\mathrm{T}}\mathbf{t} + \mathbf{Q}\boldsymbol{\tau} + \mathbf{Tq}, \\ \mathbf{E}^{\mathrm{I}} &= \mathbf{S}^{\mathrm{T}}\mathbf{t} + \mathbf{T}^{\mathrm{T}}\boldsymbol{\tau} + \mathbf{Uq}, \end{aligned} \tag{52}$$

where

$$\begin{aligned} \sigma_m &= \sigma_{ij} \text{ for } m = 1-6;\ i,j = 1, 2, 3, \\ \varepsilon_m &= \varepsilon_{ij} \text{ for } i = j,\ m = 1, 2, 3;\ \varepsilon_m = 2\varepsilon_{ij} \text{ for } i \neq j,\ m = 4, 5, 6, \\ 2\omega_{32} &= \omega_1,\ 2\omega_{13} = \omega_2,\ 2\omega_{21} = \omega_3,\ \tau_{32} = \tau_1,\ \tau_{13} = \tau_2,\ 2\tau_{21} = \tau_3, \\ P_{ijkl} &= P_{mn} \text{ when } m \text{ and } n \text{ are } 1, 2, \text{ or } 3, \\ 2P_{ijkl} &= P_{mn} \text{ when either } m \text{ or } n \text{ are } 4, 5, 6, \\ 4P_{ijkl} &= P_{mn} \text{ when both } m \text{ and } n \text{ are } 4, 5, 6, \\ 4Q_{ijkl} &= Q_{st} \text{ where } s \text{ and } t \text{ are } 1, 2, 3, \\ 2R_{ijkl} &= R_{ms} \text{ when } m \text{ is } 1, 2, 3;\ 4R_{ijkl} = R_{ms} \text{ when } m \text{ is } 4, 5, 6, \\ S_{ijk} &= S_{mk} \text{ when } m \text{ is } 1, 2, 3;\ 2S_{ijk} = S_{mk} \text{ when } m \text{ is } 4, 5, 6, \\ 2T_{ijk} &= T_{sk} \text{ where } s \text{ is } 1, 2, 3. \end{aligned} \tag{53}$$

Table 1. Rotational stiffness for various aspect ratios (isotropic, $\nu = 0.25$)

a_3/a_1	b^*_{11}/μ	b^*_{33}/μ	M	N
0.01	96.23	128.9	600	8
0.1	10.39	14.36	50	8
0.2	5.734	8.015	32	8
0.4	3.589	4.856	16	8
0.6	3.055	3.815	8	8
0.8	2.937	3.302	6	8
1.0	3.000	3.000	2	8
2.0	4.324	2.420	10	8
4.0	9.079	2.163	22	8
8.0	23.87	2.058	38	8
10.0	33.54	2.041	56	8
100.0	1649.0	2.000	420	8

Table 2. Material properties of a piezoelectric ceramic PZT-6B

Elastic stiffness (10^{10} N m^{-2})					Piezoelectric coefficients (C m^{-2})			Dielectric constants (10^{-10} F m^{-1})	
C_{11}	C_{33}	C_{44}	C_{12}	C_{13}	e_{31}	e_{33}	e_{15}	κ_{11}	κ_{33}
16.8	16.3	2.71	6.0	6.0	−0.9	7.1	4.6	36	34

Table 3. Rotational stiffness for various aspect ratios (PZT-6B)

a_3/a_1	b^*_{11} (10^{10} N m^{-2})	b^*_{33} (10^{10} N m^{-2})	M	N
0.01	389.343	495.836	1500	10
0.1	41.2636	57.5714	141	10
0.2	22.3568	33.3152	67	10
0.4	13.6520	21.3031	53	10
0.6	11.5055	17.3789	22	8
0.8	11.0816	15.4621	15	8
1.0	11.4123	14.3403	11	8
2.0	17.5450	12.2315	27	8
4.0	39.8942	11.3340	54	9
6.0	71.5111	11.0894	84	9
8.0	111.228	10.9847	113	9
10.0	158.511	10.9294	125	9
100.0	8358.66	10.8025	1300	9

The tensors **P**, **R**, **S**, **Q**, **T** and **U** are then represented by (6×6), (6×3), (6×3), (3×3), (3×3) and (3×3) matrices, respectively. The inverse of (52) is then expressed as

$$\begin{aligned}\mathbf{t} &= \mathbf{P}^*\boldsymbol{\varepsilon}^{\mathrm{I}}+\mathbf{R}^*\boldsymbol{\omega}^{\mathrm{I}}+\mathbf{S}^*\mathbf{E}^{\mathrm{I}},\\ \boldsymbol{\tau} &= \mathbf{R}^{*\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{I}}+\mathbf{Q}^*\boldsymbol{\omega}^{\mathrm{I}}+\mathbf{T}^*\mathbf{E}^{\mathrm{I}},\\ \mathbf{q} &= \mathbf{S}^{*\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{I}}+\mathbf{T}^{*\mathrm{T}}\boldsymbol{\omega}^{\mathrm{I}}+\mathbf{U}^*\mathbf{E}^{\mathrm{I}},\end{aligned} \tag{54}$$

or equivalently as an indicial notation (32), where the correspondence is

$$\begin{aligned}P^*_{ijkl} &= P^*_{mn} \text{ for } i,j,k,l = 1,2,3;\, m,n = 1\text{–}6,\\ Q^*_{ijkl} &= Q^*_{st} \text{ for } s,t = 1,2,3,\\ R^*_{ijkl} &= R^*_{mt} \text{ for } m = 1\text{–}6;\, t = 1,2,3,\\ S^*_{ijk} &= S^*_{mk} \text{ for } m = 1\text{–}6,\\ T^*_{ijk} &= T^*_{sk} \text{ for } s = 1,2,3.\end{aligned} \tag{55}$$

To check the validity of our procedures, we have compared our numerical results with existing analytic solutions for purely elastic cases. Table 1 lists the rotational stiffness b_{ij}^* versus the aspect ratio of the spheroid in an isotropic medium. The numbers M and N are the numbers of integration points necessary to achieve accuracy of four significant digits of the exact solutions (Walpole, 1991). In addition, we have checked our solutions with the numerical results tabulated in Zureick and Choi (1989) for transversely isotropic solids (mica schist and eclogite).

Finally, we present results for a piezoelectric ceramic PZT-6B. The material constants are recorded in Table 2 (Shindo and Ozawa, 1990); the symmetry corresponds to that of the hexagonal crystal of class 6 mm (Nye, 1957). The numerical values of the couple–rotation coefficient are given in Table 3 for both prolate and oblate spheroids. The numbers M and N are the necessary Gaussian station points to achieve convergence for six significant digits. This numerical procedure provides an efficient and accurate evaluation of the rotational stiffness for arbitrary ellipticity of the inclusion and for arbitrary anisotropy of the medium.

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APPENDIX

Equation (28) will be shown in this appendix. Suppose that an arbitrary elastic displacement v_i and electric field ϕ are established everywhere exterior to a closed surface Σ by some unspecified sources which lie entirely within Σ in an infinite homogeneous piezoelectric medium. Outside Σ, ε_{ij} and E_i are derived from continuously differentiable displacement fields v_i and potential ϕ, and the accompanying stress and electric displacement satisfy the equilibrium equations. At the remote boundary, the displacement and potential tend to constant, or even vanishing, values. In order to prove (15) and (28), we introduce the following integral identity over any ellipsoidal surface S which surrounds Σ

$$\int\begin{bmatrix} v_i \\ \phi \end{bmatrix} n_j\,\mathrm{d}S = \begin{bmatrix} K^1_{ikjl} & K^2_{ijl} \\ K^3_{kjl} & K^4_{jl} \end{bmatrix}\left\{\int\begin{bmatrix} L_{klpq} & -e_{qkl} \\ e_{lpq} & \kappa_{lq} \end{bmatrix}\begin{bmatrix} v_p \\ -\phi \end{bmatrix} n_q\,\mathrm{d}S - \int\begin{bmatrix} L_{kmpq} & -e_{qkm} \\ e_{mpq} & \kappa_{mq} \end{bmatrix} x_l \begin{bmatrix} v_{p,q} \\ -\phi_{,q} \end{bmatrix} n_m\,\mathrm{d}S\right\}. \quad \text{(A1)}$$

This identity can be verified by utilizing the properties

$$K^1_{ikjl} = -G^{1\text{E}}_{ik,jl} + k_{ik} n_j n_l, \quad K^2_{ijl} = -G^{2\text{E}}_{i,jl} + \frac{1}{p} k_{lm} d_m n_j n_l,$$

$$K^3_{kjl} = -G^{3\text{E}}_{k,jl} + \frac{1}{p} k_{km} d_m n_j n_l, \quad K^4_{jl} = -G^{4\text{E}}_{jl} + \frac{1}{p}\left(\frac{1}{p} k_{mn} d_m d_n - 1\right) n_j n_l,$$

$$K^1_{ikjl} x_l = -G^{1\text{E}}_{ik,j}, \quad K^2_{ijl} x_l = -G^{2\text{E}}_{i,j}, \quad K^3_{kjl} x_l = -G^{3\text{E}}_{k,j}, \quad K^4_{jl} x_l = -G^{4\text{E}}_{,j}, \tag{A2}$$

on the right-hand side of (A1). This leads to

$$\int \begin{bmatrix} -G^{1\text{E}}_{ik,jl} L_{klpq} - G^{2\text{E}}_{i,jl} e_{lpq} & G^{1\text{E}}_{ik,jl} e_{qkl} - G^{2\text{E}}_{i,jl} \kappa_{lq} \\ -G^{3\text{E}}_{k,jl} L_{klpq} - G^{4\text{E}}_{,jl} e_{lpq} & G^{3\text{E}}_{k,jl} e_{qkl} - G^{4\text{E}}_{,jl} \kappa_{lq} \end{bmatrix} \begin{bmatrix} v_p \\ -\phi \end{bmatrix} n_q \, \mathrm{d}S$$

$$+ \int \begin{bmatrix} \delta_{ip} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_p \\ -\phi \end{bmatrix} n_j \, \mathrm{d}S + \int \begin{bmatrix} G^{1\text{E}}_{ik,j} & G^{2\text{E}}_{i,j} \\ G^{3\text{E}}_{k,j} & G^{4\text{E}}_{,j} \end{bmatrix} \begin{bmatrix} L_{kmpq} & -e_{qkm} \\ e_{mpq} & \kappa_{mq} \end{bmatrix} \begin{bmatrix} v_{p,q} \\ -\phi_{,q} \end{bmatrix} n_m \, \mathrm{d}S. \tag{A3}$$

Now making use of the divergence theorem and equilibrium equations by converting them into volume integrals exterior to S, it is readily seen that the first and third members of (A3) cancel each other out. This completes the proof of eqn (A1). Next consider another ellipsoid V' enclosed by surface S' which is concentric and has the same shape and orientation with V, but has a larger size, $V \subset V'$. We can integrate the surface integral (A1) over the surfaces S and S' together, as they have the common coefficient $\mathbf{K}$. Applying the divergence theorem again on the terms inside the curly bracket of (A1), it can be shown that they all reduce to a column vector with components of stress and electric displacement, and consequently cancel each other out. Thus

$$\int_{V'-V} \begin{bmatrix} v_{i,j} \\ \phi_{,j} \end{bmatrix} \mathrm{d}V = 0. \tag{A4}$$

Letting V indefinitely approach V', the volume integral of the ellipsoidal homoeoid $V' - V$ can be transformed into a surface integral by a known result (Walpole, 1977)

$$\int_S w \begin{bmatrix} v_{i,j} \\ \phi_{,j} \end{bmatrix} \mathrm{d}S = 0. \tag{A5}$$

Now identifying v_i with $G^{1\text{E}}_{ik,l}$ and $\phi_{,i}$ with $G^{3\text{E}}_{i,k}$ and similarly for two others, we arrive at the results of (28). Also, this conclusion can be applicable to (15).